A second-order semi-Lagrangian particle FEM (SL-PFEM) method for the incompressible Navier-Stokes equations

B. Serván-Camas [a], J. García-Espinosa [a,b], Jonathan Colom-Cobb [a]

[a] *Centre Internacional de Mètodes Numèrics en Enginyeria (CIMNE), Division of Marine Research, Gran Capitan s/n, 08034 Barcelona, Spain*

[b] *Universitat Politècnica de Catalunya, BarcelonaTech (UPC), Campus Nàutica, Edif. NT3, C. Escar 6-8, 08039 Barcelona, Spain*

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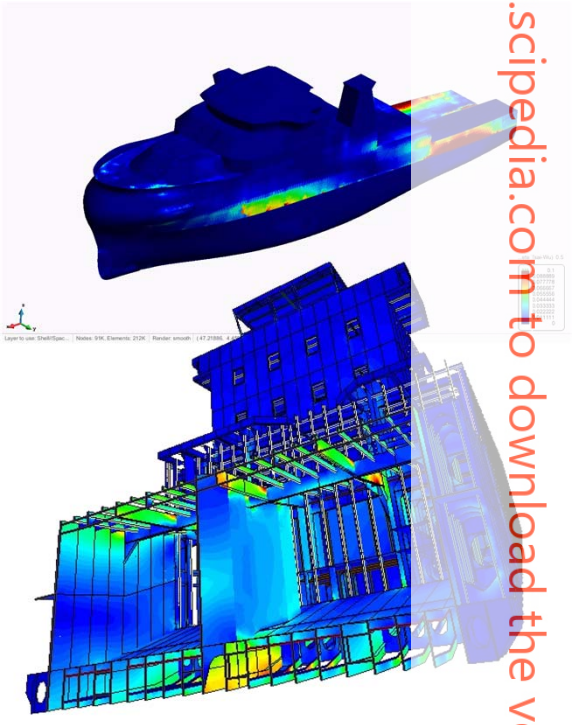
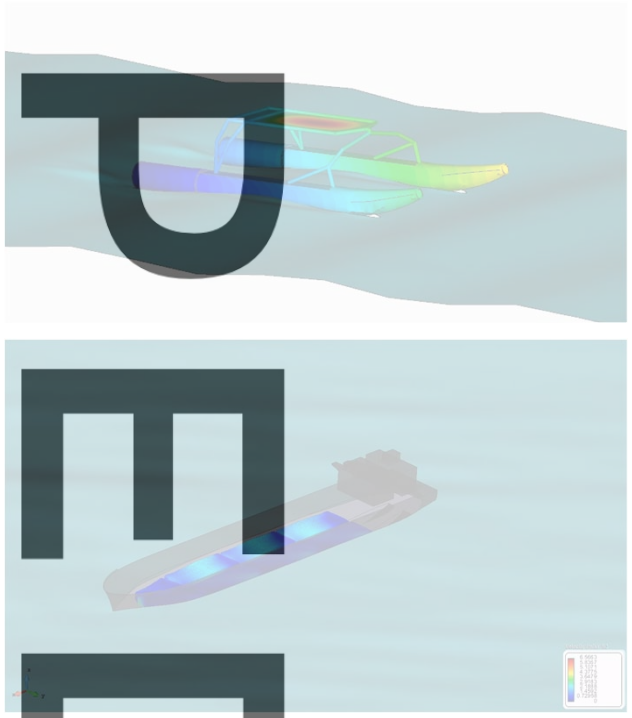
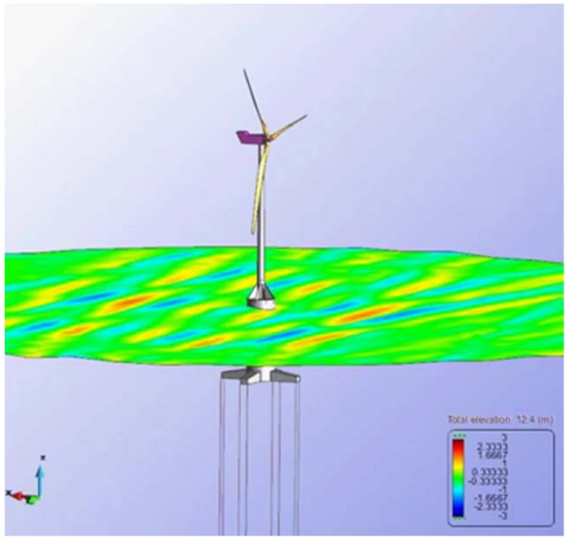


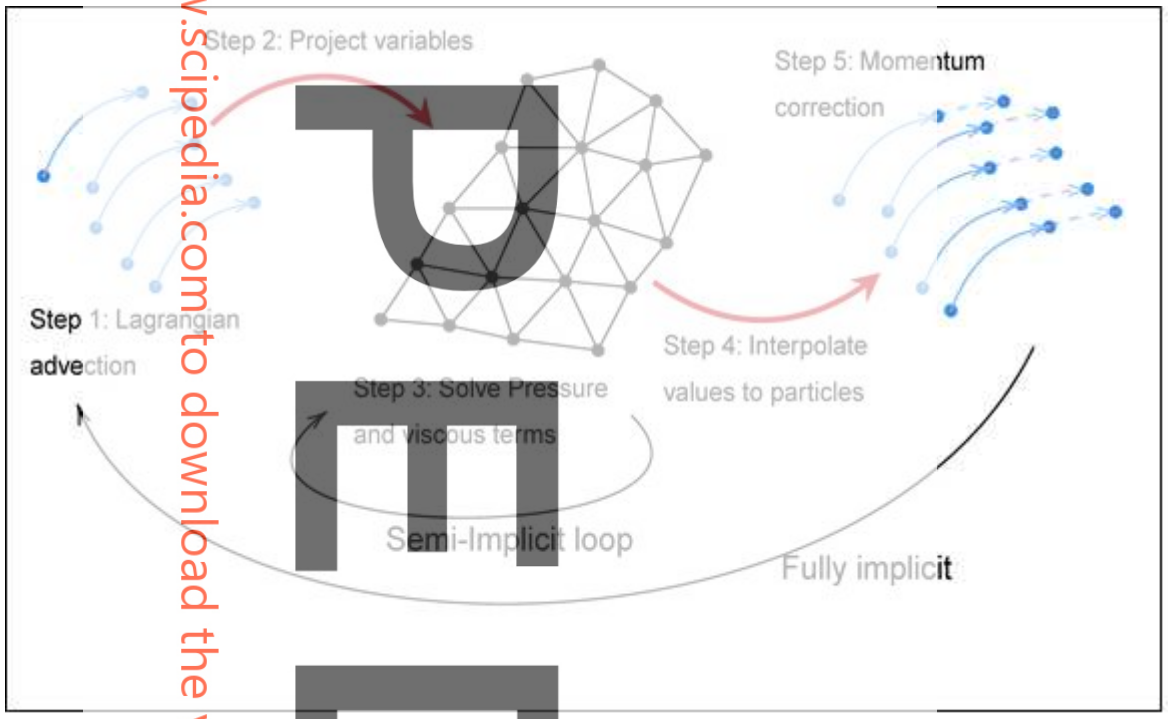




OUTLINE

Concepts of the Semi-Lagrangian particle Finite Element Method (SL-PFEM)

First introduced by: S. R. Idelsohn, N. Nigro, A. Limache, E. Oñate: Large time-step explicit integration method for solving problems with dominant convection. Computer Methods in Applied Mechanics and Engineering 217-220, 168–185 (2012). DOI 10.1016/j.cma.2011.12. 008.



SEMI-LAGRANGIAN APPROACH

SEMI-LAGRANGIAN APPROACH

Integrating the particle's equation of motion

Let $\boldsymbol{a}(\boldsymbol{x}, t)$ be an acceleration field and let $\{\lambda\}$ be a set of particles each of them identified with a label λ.

Particle's equation of motion:

$$\begin{aligned} \mathrm{d}_t \boldsymbol{U}_\lambda(t) &= \boldsymbol{A}_\lambda(t) = \boldsymbol{a}(\boldsymbol{X}_\lambda(t), t) \\ \mathrm{d}_t \boldsymbol{X}_\lambda(t) &= \boldsymbol{U}_\lambda(t) \end{aligned}$$

Velocity Verlet algorithm:

$$\begin{aligned} \boldsymbol{X}_\lambda(t^{n+1}) &= \boldsymbol{X}_\lambda(t^n) + \Delta t \boldsymbol{U}_\lambda(t^n) + \frac{\Delta t^2}{2} \boldsymbol{A}_\lambda(t^n) + O(\Delta t^3) \\ \boldsymbol{U}_\lambda(t^{n+1}) &= \boldsymbol{U}_\lambda(t^n) + \frac{\Delta t}{2} \left(\boldsymbol{A}_\lambda(t^n) + \boldsymbol{A}_\lambda(t^{n+1}) \right) + O(\Delta t^3) \end{aligned}$$

SL-PFEM for the incompressible Navier-Stokes equations

Let $\boldsymbol{u}(\boldsymbol{x},t)$ be a fluid velocity field and let´s define the acceleration field $\boldsymbol{a}(\boldsymbol{x},t)$:

$$\boldsymbol{a} = \mathrm{d_t}\boldsymbol{u} = \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\boldsymbol{\nabla}\left(\frac{P}{\rho}\right) + \nu\Delta\boldsymbol{u} + \boldsymbol{f}$$

Move particles (explicit):

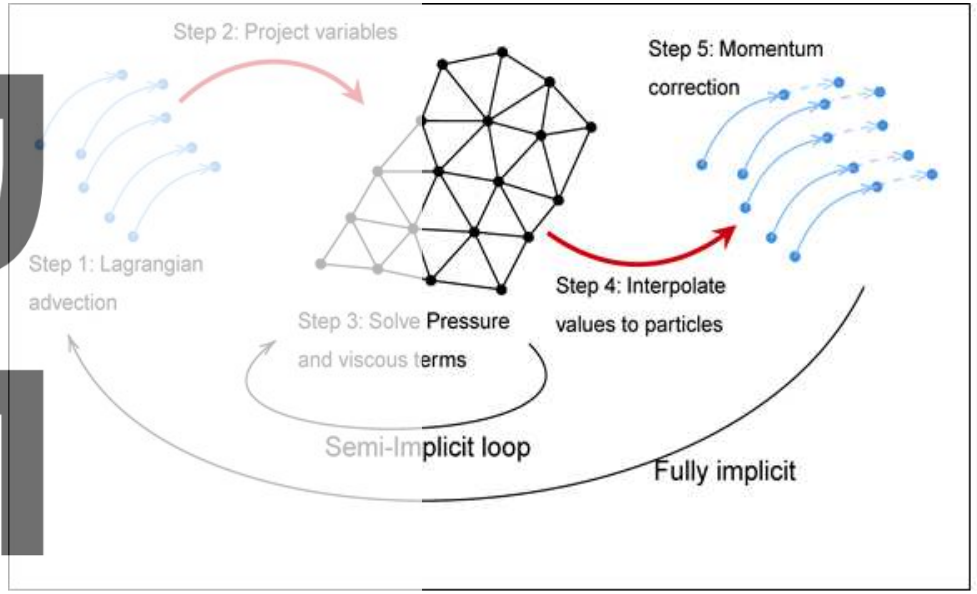
$$\boldsymbol{X}_\lambda^{n+1} = \boldsymbol{X}_\lambda^n + \Delta t\boldsymbol{u}^n\left(\boldsymbol{X}_\lambda^n + \frac{\Delta t}{2}\boldsymbol{a}^n\left(\boldsymbol{X}_\lambda^n\right)\right)$$

Update velocity (explicit):

$$\boldsymbol{U}_\lambda^{n+1/2} = \boldsymbol{U}_\lambda^n + \frac{\Delta t}{2}\boldsymbol{a}^n\left(\boldsymbol{X}_\lambda^n\right)$$

Correct velocity (implicit):

$$\boldsymbol{U}_\lambda^{n+1} = \boldsymbol{U}_\lambda^{n+1/2} + \frac{\Delta t}{2}\boldsymbol{a}^{n+1}\left(\boldsymbol{X}_\lambda^{n+1}\right)$$



SL-PFEM for the incompressible Navier-Stokes equations

Projection onto FEM mesh: $\boldsymbol{u}^{n+1}(\boldsymbol{x}) = \mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{U}_\lambda^{n+1}\}\right]$:

$$\mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{U}_\lambda^{n+1}\}\right] = \mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{U}_\lambda^{n+1/2}\}\right] + \frac{\Delta t}{2}\mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{a}^{n+1}(\boldsymbol{X}_\lambda^{n+1})\}\right]$$

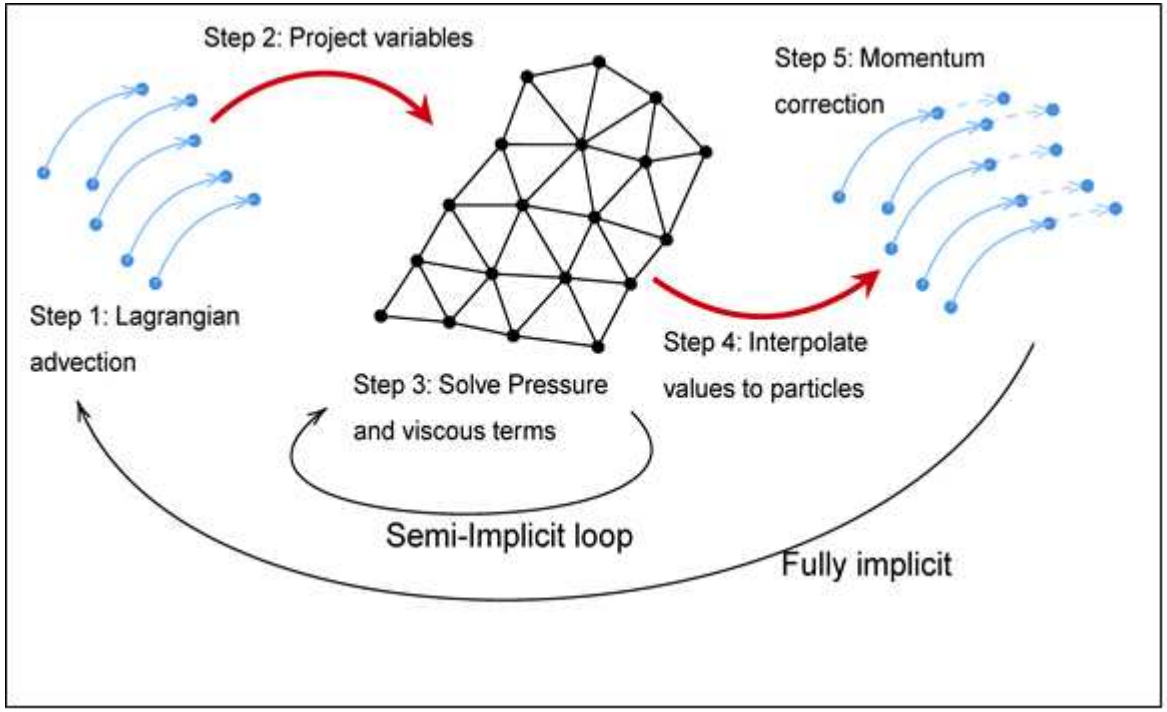
$$\boldsymbol{u}^{n+1}(\boldsymbol{x}) = \boldsymbol{u}_{\mathcal{P}}^{n+1/2} + \frac{\Delta t}{2}\mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{a}^{n+1}(\boldsymbol{X}_\lambda^{n+1})\}\right]$$

Coherence condition: $\boldsymbol{a}^{n+1}(\boldsymbol{x}) = \mathcal{P}_{\{\lambda\}}^{n+1}\left[\{\boldsymbol{a}^{n+1}(\boldsymbol{X}_\lambda^{n+1})\}\right]$

$$\frac{\boldsymbol{u}^{n+1}(\boldsymbol{x}) - \boldsymbol{u}_{\mathcal{P}}^{n+1/2}(\boldsymbol{x})}{\Delta t} = \frac{1}{2}\boldsymbol{a}^{n+1}(\boldsymbol{x})$$

SL-PFEM

SL-PFEM for the incompressible Navier-Stokes equations



Remark: the coherence condition makes it unneccesary to iterate at the outter implicit loop.

SL-PFEM

Semi-Lagrangian approach for the incompressible Navier-Stokes equations

Eulerian description (Stokes' type equation):

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}_{\mathcal{P}}^{n+1/2}}{\Delta t} = \frac{1}{2}\underbrace{\left(-\boldsymbol{\nabla}\left(\frac{P^{n+1}}{\rho}\right) + \nu\Delta\boldsymbol{u}^{n+1} + \boldsymbol{f}^{n+1}\right)}_{\boldsymbol{a}^{n+1}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^{n+1}=0$$

Solved using FEM implicit scheme inspired in the fractional step method.

SL-PFEM

Semi-Lagrangian approach for the incompressible Navier-Stokes equations

FEM scheme inspired in the fractional step method):

$\nu = 0\ (Non - iterative)$:

$$\bar{\bar{\boldsymbol{L}}}\left(\frac{1}{\rho}P^{n+1}\right)=\frac{2}{\Delta t}\bar{\bar{\boldsymbol{D}}}\boldsymbol{u}_{\mathcal{P}}^{\mathrm{n+1/2}}$$

$$\frac{1}{\Delta t}\bar{\bar{\boldsymbol{I}}}\boldsymbol{u}_{\mathcal{P}}^{\mathrm{n+1/2}}-\frac{1}{2}\bar{\bar{\boldsymbol{G}}}\left(\frac{1}{\rho}P^{n+1,i}\right)+\frac{1}{2}\bar{\bar{\boldsymbol{I}}}\boldsymbol{f}^{n+1}=\left(\frac{1}{\Delta t}\bar{\bar{\boldsymbol{I}}}-\frac{\nu}{2}\bar{\bar{\boldsymbol{L}}}\right)\boldsymbol{u}^{\mathrm{n+1,i+1}}$$

$\nu \neq 0\ (iterative)$:

$$\bar{\bar{\boldsymbol{L}}}\left(\frac{1}{\rho}P^{n+1,i}\right)=-\bar{\bar{\boldsymbol{D}}}\bar{\bar{\boldsymbol{G}}}\left(\frac{P^{n+1,i}}{\rho}\right)+\frac{2}{\Delta t}\bar{\bar{\boldsymbol{D}}}\boldsymbol{u}^{\mathrm{n+1,i+1}}$$

SL-PFEM

Projection

Minimization of the least square error (LSE).

$\{\Psi_\lambda\}$: set of particles' values $\xrightarrow{Projection} \mathcal{P}[\{\Psi_\lambda\}] = \{\psi_c^*\}$ projected nodal values

Interpolated-projected values on particles: $\psi_h(\boldsymbol{X}_\lambda) = \sum_c N^c(\boldsymbol{X}_\lambda)\psi_c^*$

Square error: $\epsilon_\psi = \sum_\lambda (\psi_h(\boldsymbol{X}_\lambda) - \Psi_\lambda)^2$

LSE: $\frac{\partial \epsilon_\psi}{\partial \psi_b^*} = 0 \rightarrow \sum_\lambda \left(\sum_c N^b(\boldsymbol{X}_\lambda) N^c(\boldsymbol{X}_\lambda) \psi_c^* \right) = \sum_\lambda N^b(\boldsymbol{X}_\lambda) \Psi_\lambda$

Fulfils the coherence condition naturally

$$\Psi_\lambda = \psi_h(\boldsymbol{X}_\lambda) \rightarrow \epsilon_\psi = 0$$

SL-PFEM

Advantages of the SL-PFEM:

1. Explicit Lagrangian scheme for momentum equation with no stability restriction based on Courant number and low numerical erosion.
    - The time step is not lmited by numerical stability. However, time step must be low enough to capture the physics. For instance, the time step must be small enough so that the particles' trajectories are accurate enough.

2. Explicit Lagrangian operations ("move particles" and "update velocity") are embarrassingly parallel.

3. Particles can carry the information of any type of intrinsic variables.

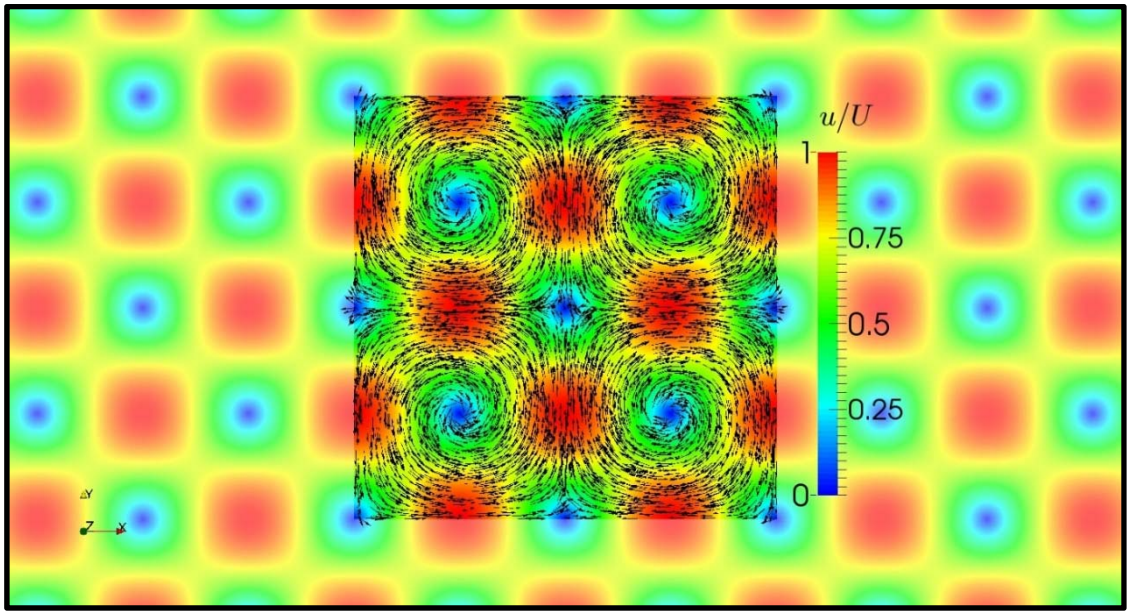
4. FEM is very suitable to solve Stokes type equations.

VERIFICATION AND CONVERGENCE ANALYSES

TAYLOR-GREEN VORTEX

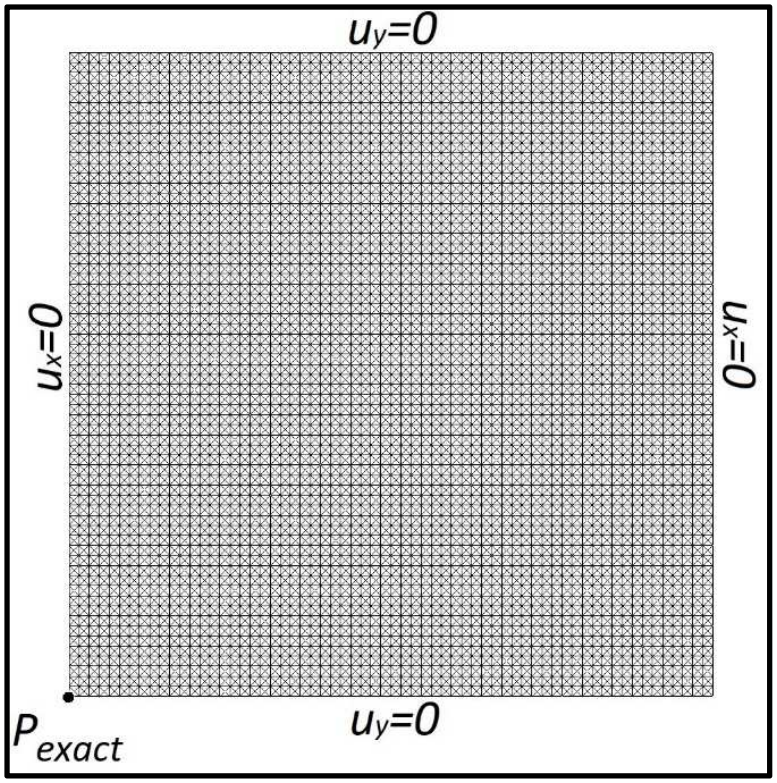
Taylor-Green vortex solution

$$u_x(x,y,t) = -\sin(x)\cos(y)\,e^{-2\nu t}$$
$$u_y(x,y,t) = +\cos(x)\sin(y)\,e^{-2\nu t}$$
$$P(x,y,t) = \frac{1}{4}[\cos(2x)+\cos(2y)]e^{-4\nu t}$$

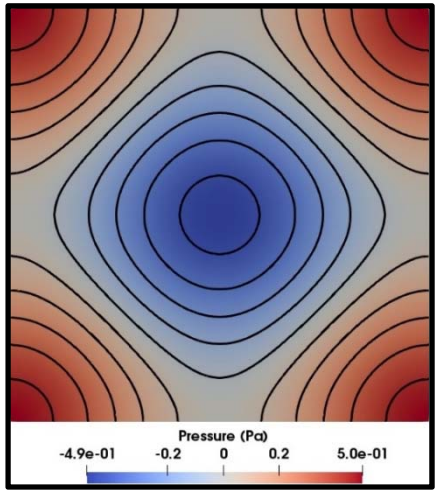


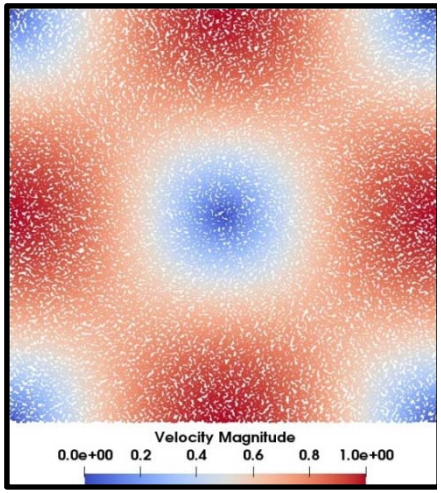
TAYLOR-GREEN VORTEX

Taylor-Green vortex decay



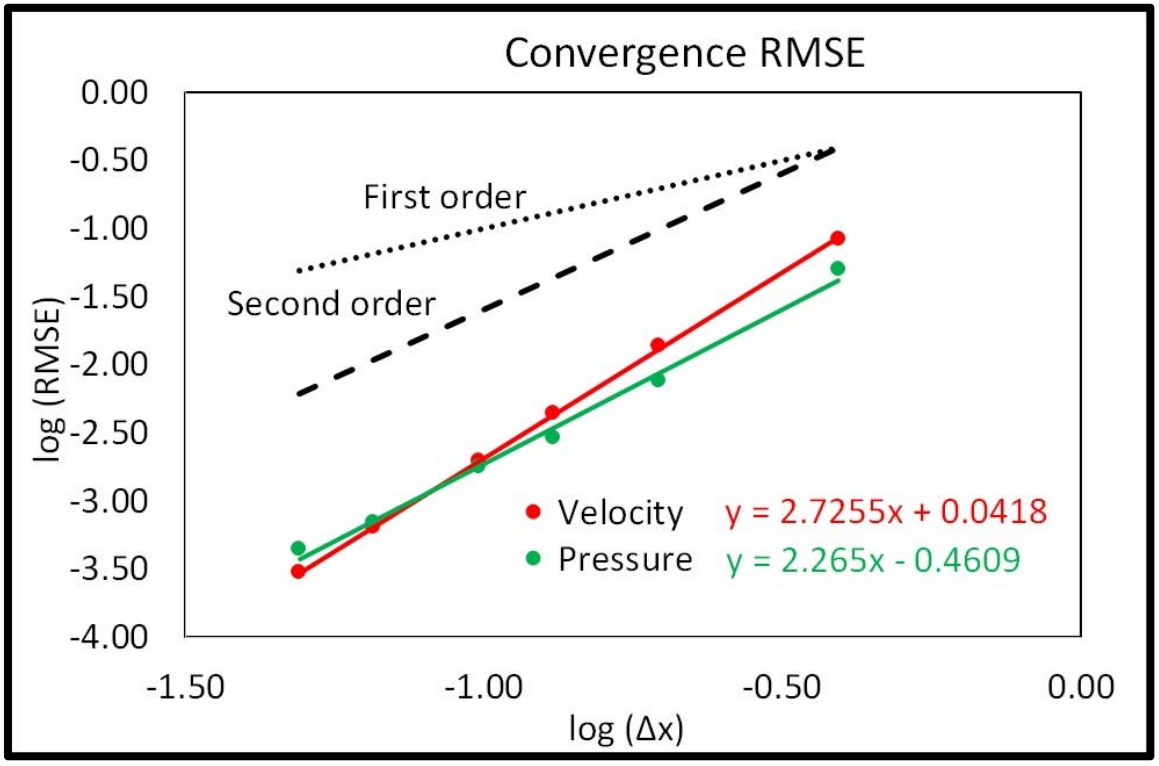
Particulars	
Maximum velocity (m/s)	1
Reynolds number	**3140**
Domain size (mxm)	$(0, \Pi)$ x$(0, \Pi)$
Courant number	**0. 5**
Simulation time (s)	10





CONVERGENCE ANALYSIS

Taylor-Green vortex decay



Case	Δx(m)	Δt(s)	N Δt
8x8	$\pi/8$	0.2	50
16x16	$\pi/16$	0.1	100
24x24	$\pi/24$	0.0667	150
32x32	$\pi/32$	0.05	200
48x48	$\pi/48$	0.0333	300
64x64	$\pi/64$	0.025	400

CONVERGENCE ANALYSIS

Steady-state Taylor Green Vortex

Mass forces:

$f_x(x,y) = -2\nu \sin(x)\cos(y)$

$f_y(x,y) = 2\nu \cos(x)\sin(y)$

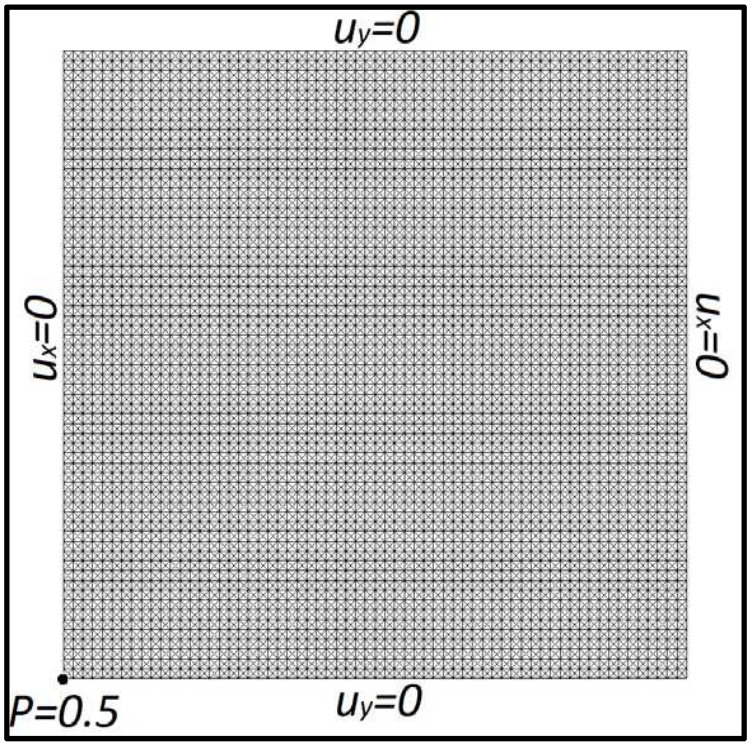
Analytical solution:

$u_x(x,y) = -\sin(x)\cos(y)$

$u_y(x,y) = \cos(x)\sin(y)$

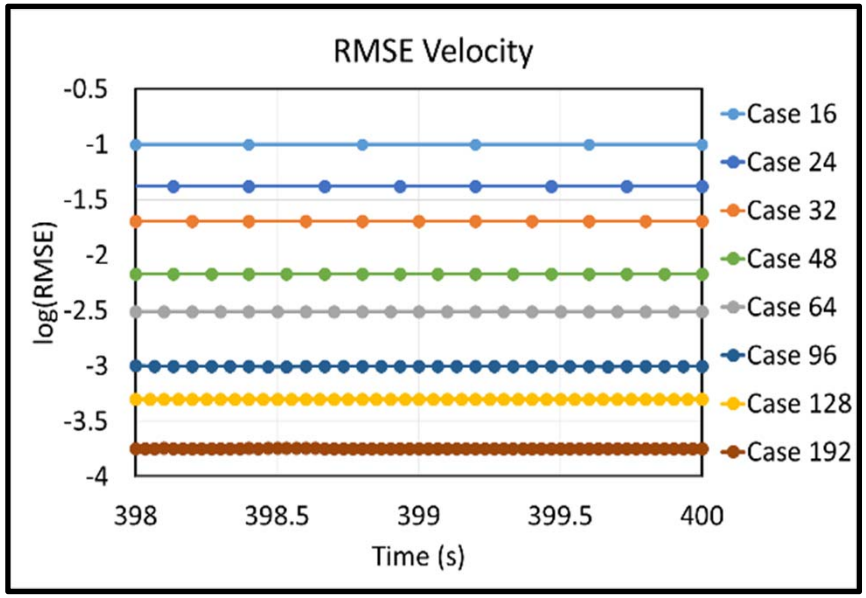
$P(x,y) = 0.25[\cos(2x) + \cos(2y)]$

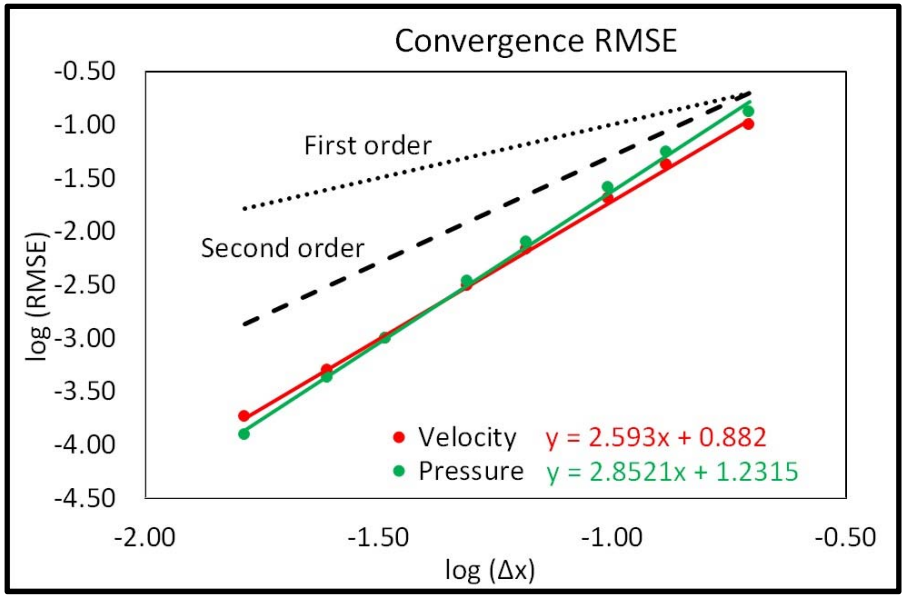
Particulars	
Maximum velocity (m/s)	1
Reynolds number	**314**
Domain size (mxm)	$(0,\Pi)$ x$(0,\Pi)$
Courant number	**2.04**
Simulation time (s)	400



CONVERGENCE ANALYSIS

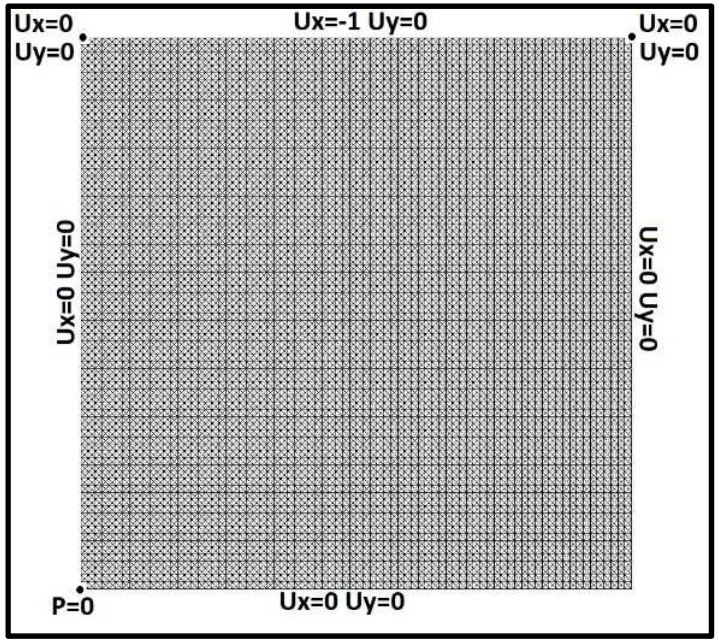
Case	Δx(m)	Δt(s)	N Δt
16x16	$\pi/16$	0.4	1000
24x24	$\pi/24$	0.2667	1500
32x32	$\pi/32$	0.2	2000
48x48	$\pi/48$	0.1333	3000
64x64	$\pi/64$	0.1	4000
96x96	$\pi/96$	0.0667	6000
128x128	$\pi/128$	0.05	8000
192x192	$\pi/192$	0.0333	12000





LID DRIVEN CAVITY FLOW

Lid velocity V	$-1m/s$
Reynolds number Re	1000
Domain size	$1m\ x\ 1m$
Domain discretization	$80x80$
Number of elements	25600
Number of nodes	12961
Particles per element	3
Mesh size $\Delta x = \Delta y$	0.0125
Time step Δt	0.1
Courant number	$\mathbf{8}$
Simulation time	$10^5 \Delta t$
Sampling time	$10^2 \Delta t$

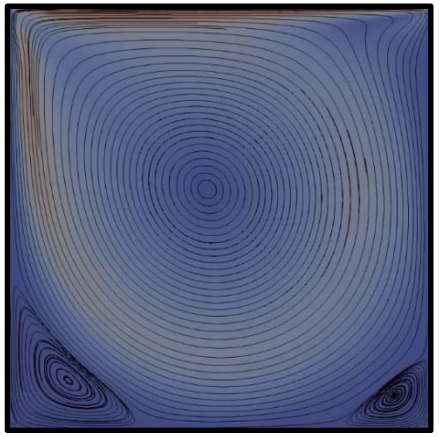


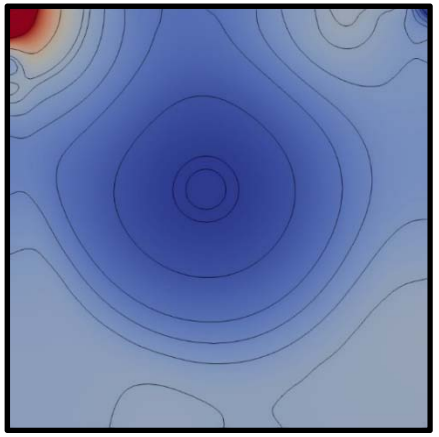
LID DRIVEN CAVITY FLOW

Velocity-streamlines

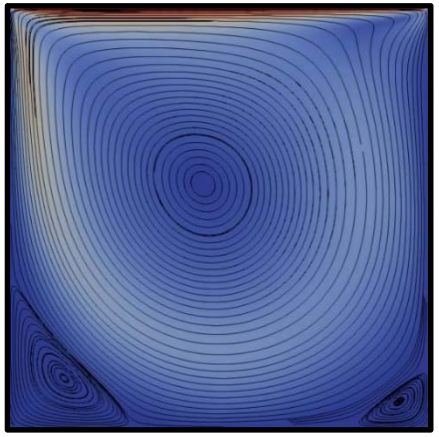
Pressure

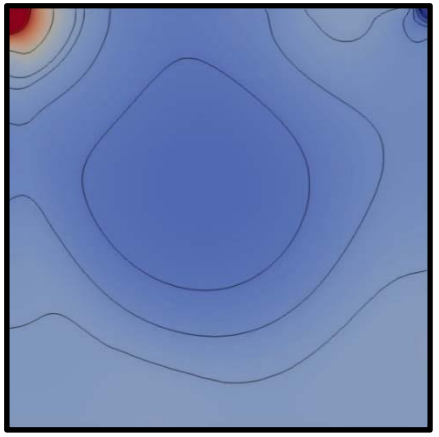
Second order velocity
Verlet SL-PFEM





First order
Euler SL-PFEM

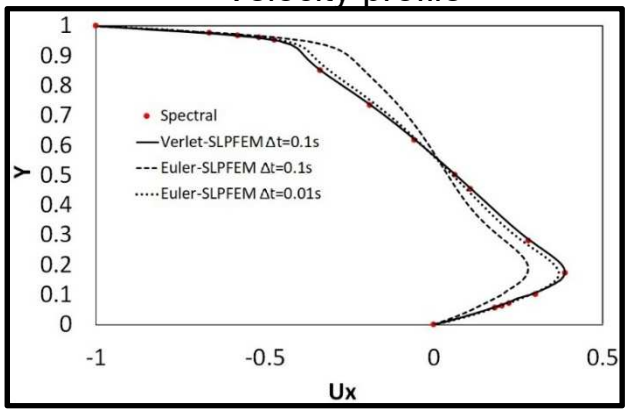




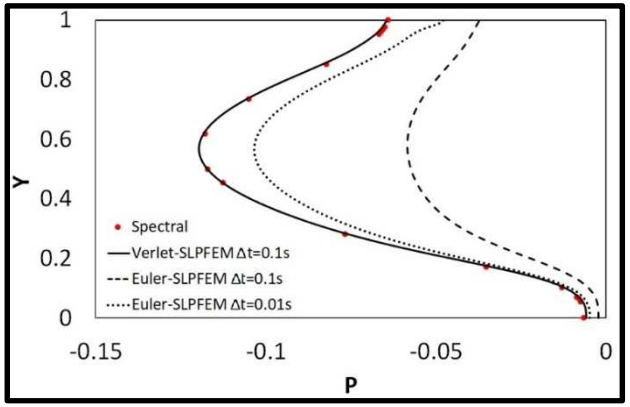
LID DRIVEN CAVITY FLOW

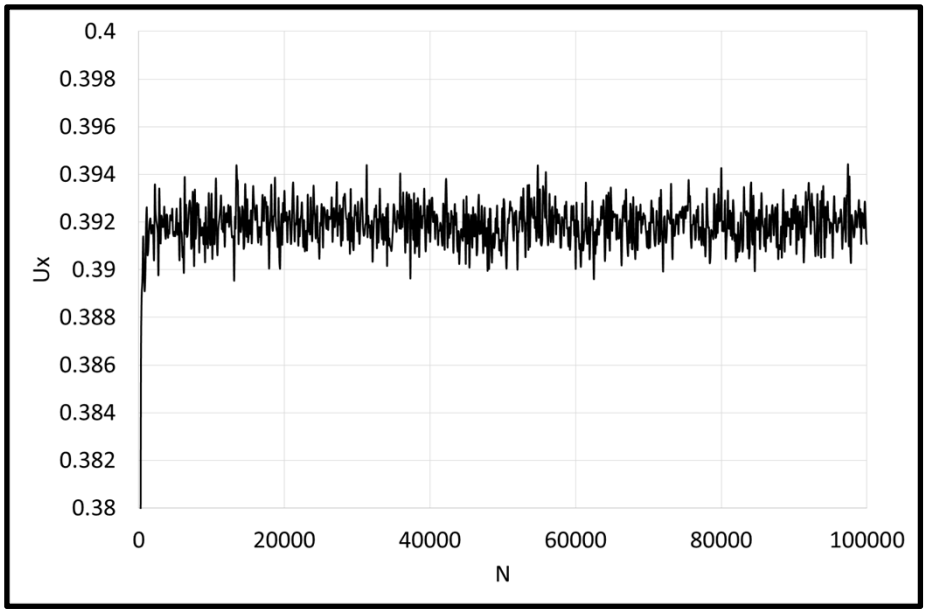
Horizontal velocity and pressure profiles at the mid-section x=0.5. Solid line: Second Order Verlet-SLPFEM. Dash line: First Order in time Euler-SLPFEM. Red dots: spectral method (O. Botella and R. Peyret. Benchmark spectral results on the lid-driven cavity flow. Computers & Fluids Vol. 27, No. 4, pp. 421-433, 1998)

Velocity profile



Pressure profile



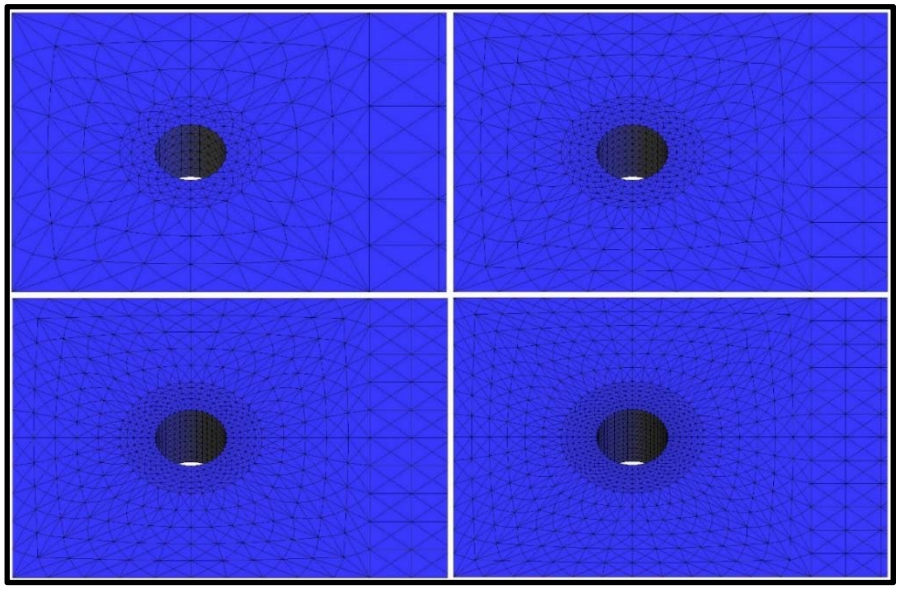


Horizontal velocity evolution at (x,y)=(0.5,0.175) for the second-order Verlet -SLPFEM

3D FLOW PAST A CYLINDER

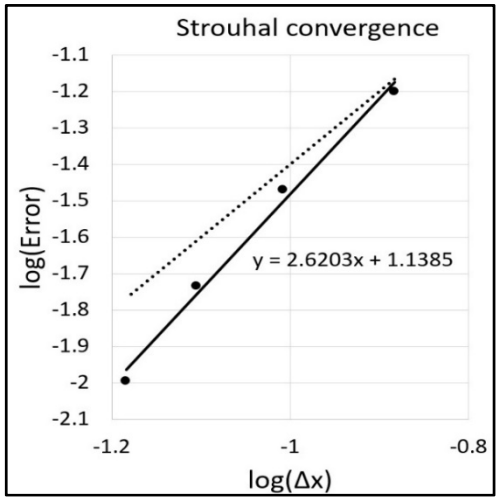
Mesh	Δx(m)	Δt(s)	**N tetras**	**N Nodes**	**N Δt**
1	0.1333	0.0667	62208	14487	1500
2	0.1	0.05	147456	33412	2000
3	0.08	0.04	288000	64185	2500
4	0.0667	0.0333	497664	109686	3000

Particulars	
Cylinder diameter (m)	1
Cylinder height (m)	1
Inlet velocity (m/s)	1
Reynolds number	200
Courant number	2

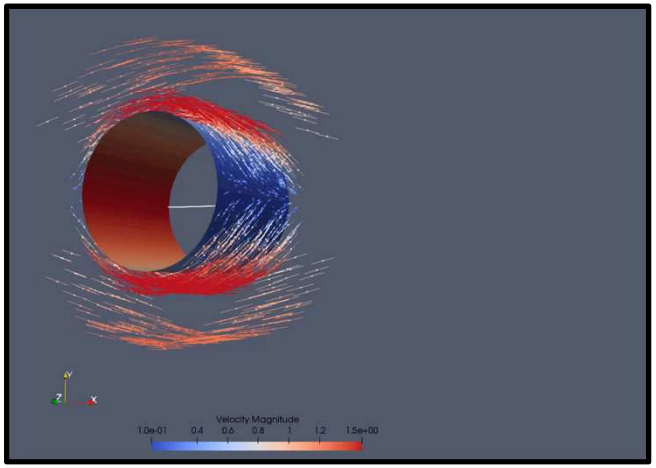


3D FLOW PAST A CYLINDER

Case	St	Error St	C_L
1	0.133	0.0630	0.177
2	0.162	0.0339	0.255
3	0.178	0.0184	0.246
4	0.186	0.0101	0.258
Ref. [*]	0.196		



CPU time in seconds per time step (mesh 2)			SL-PFEM		FEM
Particles	Move		0.092	0.602	
	Projector	Assembly	0.363		
		Solver	0.147		
Velocity solver			0.125		1.532
Pressure solver			0.339		0.320
Total			1.066		1.852



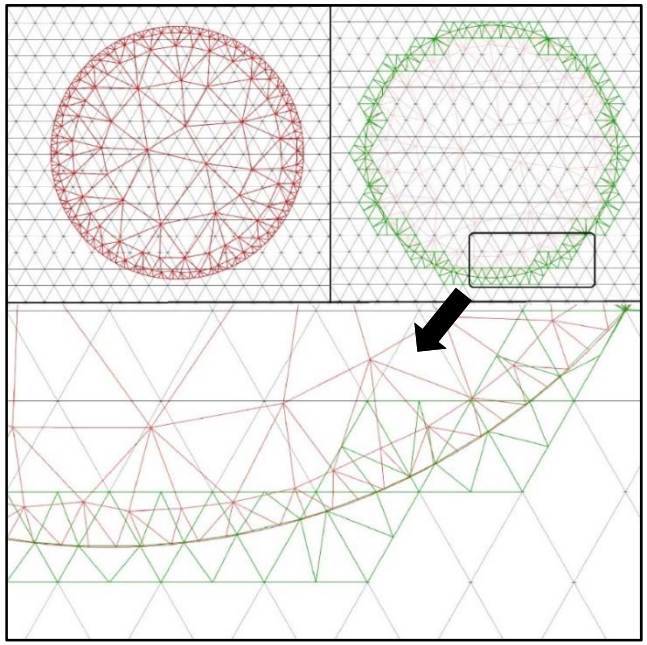
* H. Jiang and L. Cheng. Strouhal Reynolds number relationship for flow past a circular cylinder. J. Fluid Mech. (2017), vol. 832, pp. 170-188

ENRICHED SL-PFEM

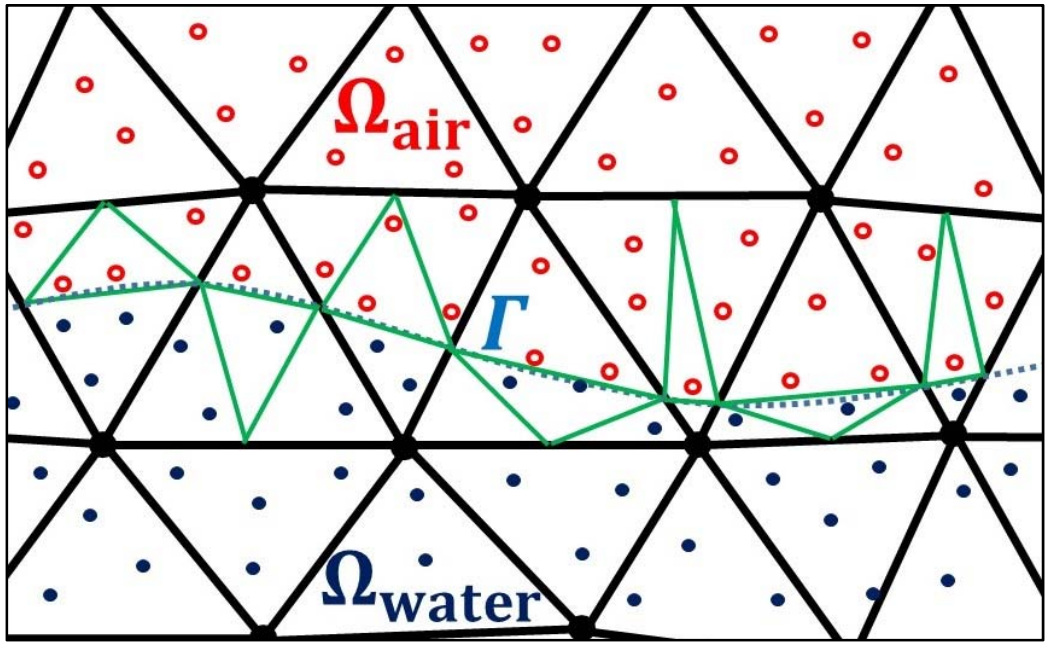
ENRICHMENT

Based on local refinement and the introduction of new degrees of freedom at the interface

Fluid-Solid:



Free Surface:

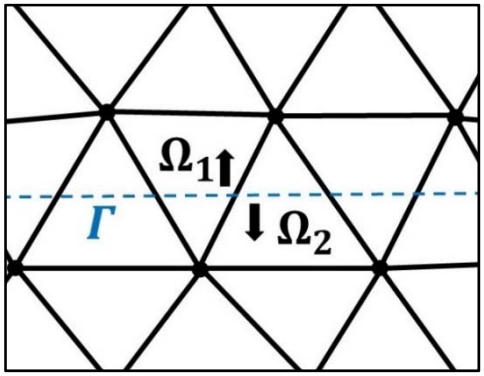


ENRICHMENT

Pressure Enrichment:

Governing Eq: $\Delta P_f = \rho_f \nabla \cdot \boldsymbol{u}_{\mathcal{P}}$

BC: $\frac{1}{\rho_{f1}} \nabla P_{f1} \cdot \boldsymbol{n}_\Gamma = \frac{1}{\rho_{f2}} \nabla P_{f2} \cdot \boldsymbol{n}_\Gamma \; on \; \Gamma$



Variational formulation:

$$\int_{\Omega_f} \omega \, \Delta P_f d\Omega_f = -\int_{\Omega_f} \nabla\omega \nabla P_f d\Omega + \int_{\Gamma_f} \omega\left(\nabla P_f \cdot \boldsymbol{n}\right) d\Gamma_f = \rho_f \int_{\Omega_f} \omega(\nabla \cdot \boldsymbol{u}_{\mathcal{P}}) \, d\Omega_f$$

FEM matrix system of equations: $\bar{\bar{\boldsymbol{K}}}_f \boldsymbol{P}_f = \bar{\bar{\boldsymbol{D}}} \cdot \boldsymbol{u}_{\mathcal{P}} - \left(\bar{\bar{\boldsymbol{D}}}_f + \bar{\bar{\boldsymbol{D}}}_f^T\right) \cdot \bar{\bar{\boldsymbol{G}}}_f \boldsymbol{P}_f$

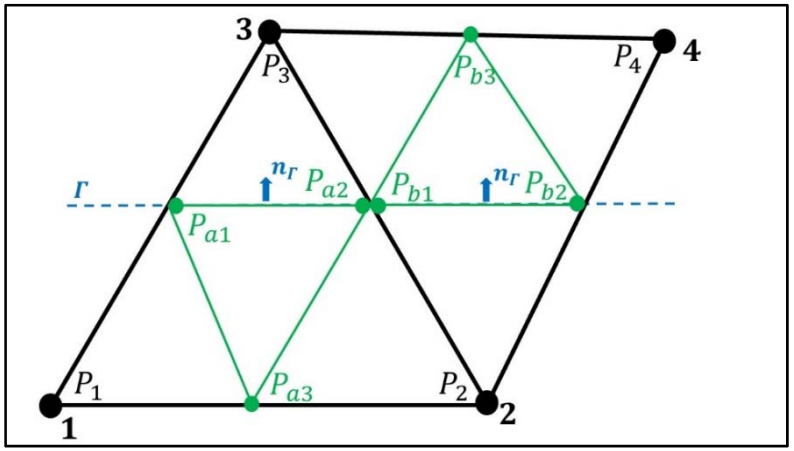
where

$$\bar{\bar{\boldsymbol{K}}}_f = -\frac{1}{\rho_f}\left[\int_{\Omega_f} \nabla N^i \, \nabla N^j d\Omega_f\right] \qquad \bar{\bar{\boldsymbol{G}}}_f = \frac{1}{\rho_f} \int_{\Omega_f} N^i \, \nabla N^j d\Omega_f \Big/ \int_{\Omega_f} N^i d\Omega_f$$

$$\bar{\bar{\boldsymbol{D}}}_f = \int_{\Omega_f} N^i \, \nabla N^j d\Omega_f \qquad \bar{\bar{\boldsymbol{D}}}_f^T = \int_{\Omega_f} \nabla N^i \, N^j d\Omega_f$$

$$\left(\bar{\bar{\boldsymbol{D}}}_f + \bar{\bar{\boldsymbol{D}}}_f^T\right) \cdot \bar{\bar{\boldsymbol{G}}}_f \boldsymbol{P}_f = \left[\int_{\Gamma_f} N^i N^j \boldsymbol{n} d\Gamma_f\right] \cdot \bar{\bar{\boldsymbol{G}}}_f \boldsymbol{P}_f$$

ENRICHMENT



Enriched matrix:

$$\begin{bmatrix} K_{11} & & & KA_{11} & 0 & KA_{13} & 0 & 0 & 0 \\ & K_{22} & & 0 & KA_{22} & KA_{23} & KB_{21} & KB_{22} & 0 \\ & & K_{33} & KA_{31} & 0 & KA_{33} & KB_{31} & 0 & KB_{33} \\ & & & 0 & 0 & 0 & 0 & KB_{42} & KB_{43} \\ \overline{\overline{KA}}^T & & & A_{11} & A_{12} & A_{13} & \multicolumn{3}{c}{\multirow{3}{*}{$\bar{\bar{0}}$}} \\ & & & A_{21} & A_{22} & A_{23} & & & \\ & & & A_{31} & A_{32} & A_{33} & & & \\ \overline{\overline{KB}}^T & & & \multicolumn{3}{c}{\multirow{2}{*}{$\bar{\bar{0}}$}} & B_{11} & B_{12} & B_{13} \\ & & & & & & B_{21} & B_{22} & B_{23} \\ & & & & & & B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Interpolate:

$$\nabla \bar{P}_{a2} = \nabla \bar{P}_{b1} = Interp(\nabla \bar{P}_2, \nabla \bar{P}_3)$$

$$\boldsymbol{u}_{a2} = \boldsymbol{u}_{b1} = Interp(\boldsymbol{u}_2, \boldsymbol{u_3})$$

Enriched system:

$$\begin{bmatrix} \bar{\bar{\boldsymbol{I}}}_{\boldsymbol{K}} & \overline{\overline{\boldsymbol{KA}}} & \overline{\overline{\boldsymbol{KB}}} \\ \overline{\overline{\boldsymbol{KA}}}^{\boldsymbol{T}} & \bar{\bar{\boldsymbol{A}}} & \bar{\bar{\boldsymbol{0}}} \\ \overline{\overline{\boldsymbol{KB}}}^{\boldsymbol{T}} & \bar{\bar{\boldsymbol{0}}} & \bar{\bar{\boldsymbol{B}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{P}_A \\ \boldsymbol{P}_B \end{bmatrix} = \begin{bmatrix} \boldsymbol{rhs} \\ \boldsymbol{rhs}_A \\ \boldsymbol{rhs}_B \end{bmatrix}$$

Collapse enriched nodes: $\boldsymbol{P}_A = \bar{\bar{\boldsymbol{A}}}^{-1}\left(\boldsymbol{rhs}_A - \overline{\overline{KA}}^T \boldsymbol{P}\right) \qquad \boldsymbol{P}_B = \bar{\bar{\boldsymbol{B}}}^{-1}\left(\boldsymbol{rhs}_B - \overline{\overline{\boldsymbol{KB}}}^T \boldsymbol{P}\right)$

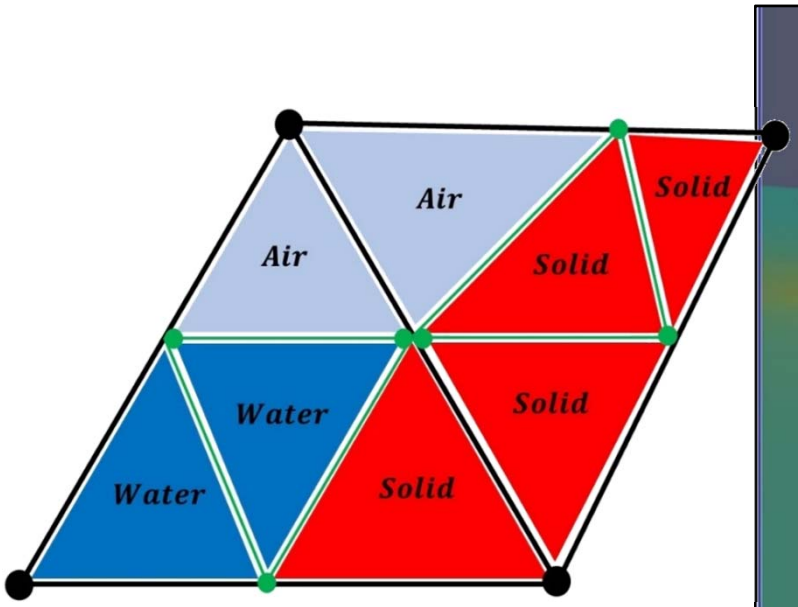
New pressure system: $\left[\bar{\bar{\boldsymbol{I}}}_{\boldsymbol{K}} - \overline{\overline{\boldsymbol{KA}}}\,\bar{\bar{\boldsymbol{A}}}^{-1}\overline{\overline{\boldsymbol{KA}}}^{\boldsymbol{T}} - \overline{\overline{\boldsymbol{KB}}}\,\bar{\bar{\boldsymbol{B}}}^{-1}\overline{\overline{\boldsymbol{KB}}}^{T}\right][\boldsymbol{P}] = \left[\boldsymbol{rhs} - \overline{\overline{\boldsymbol{KA}}}\,\bar{\bar{\boldsymbol{A}}}^{-1}\boldsymbol{rhs}_A - \overline{\overline{\boldsymbol{KB}}}\,\bar{\bar{\boldsymbol{B}}}^{-1}\boldsymbol{rhs}_A\right]$

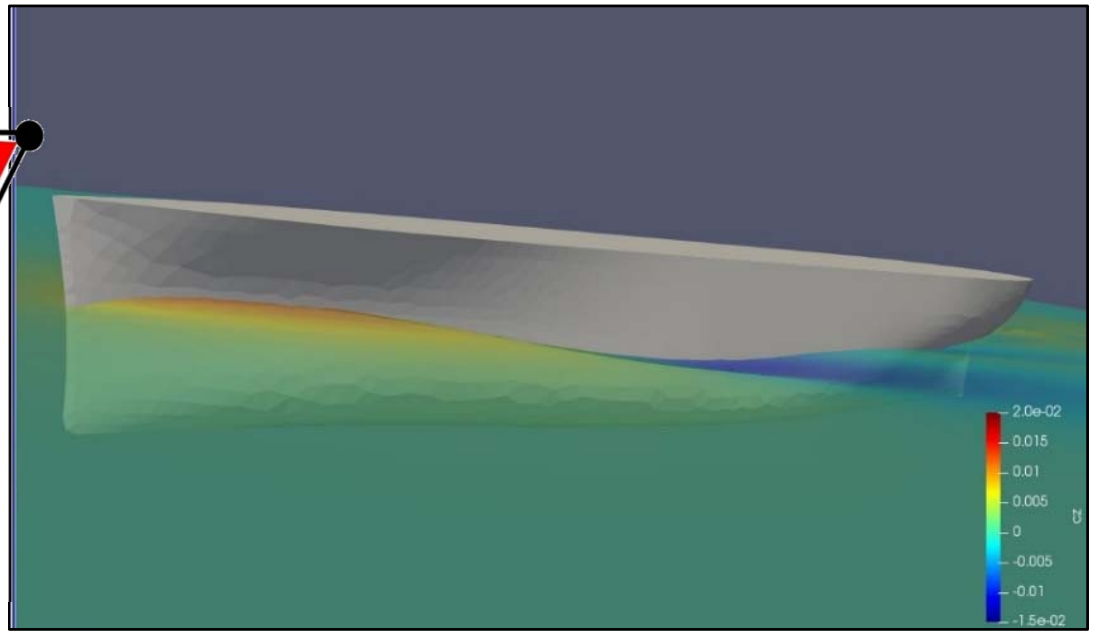
ENRICHMENT

Velocity Enrichment for solid fluid interface:

New velocity system: $\left[\bar{\bar{I}}_K\right][\boldsymbol{U}] = \left[\boldsymbol{rhs} - \overline{\overline{KA}}\,\boldsymbol{U}_{\mathrm{A}} - \overline{\overline{KB}}\,\boldsymbol{U}_B\right]$

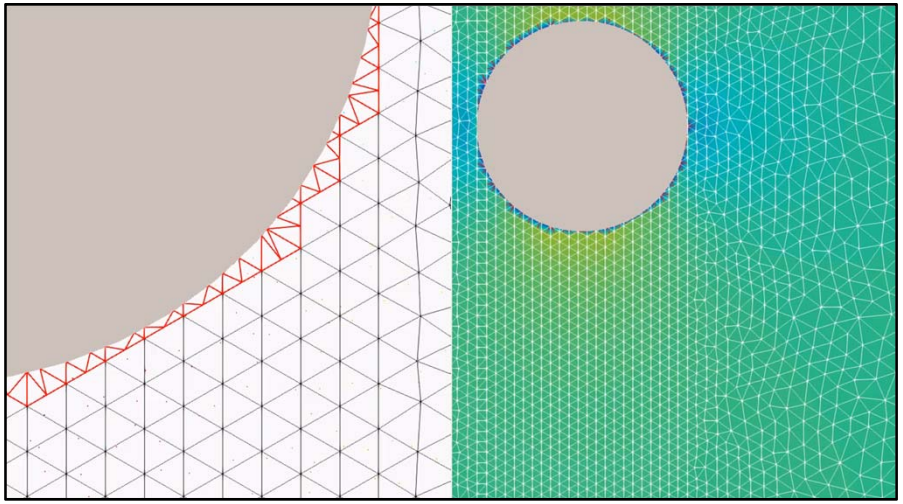
Water-Air-Solid interfaces:



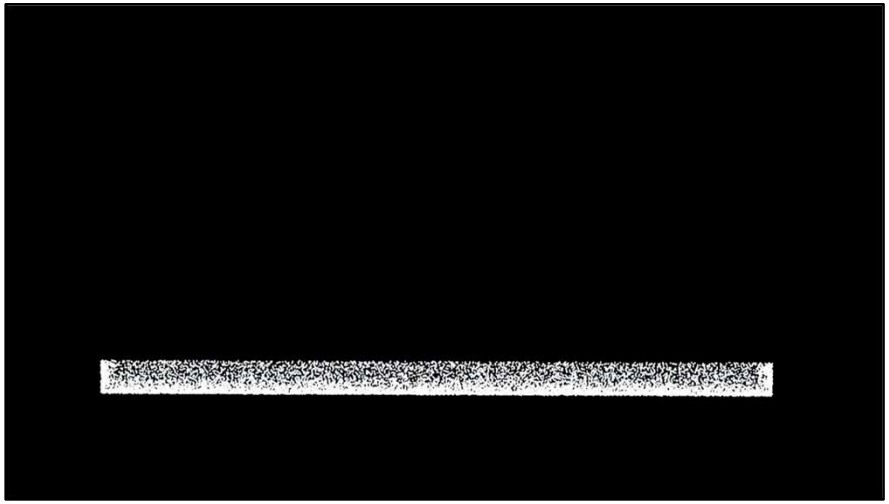


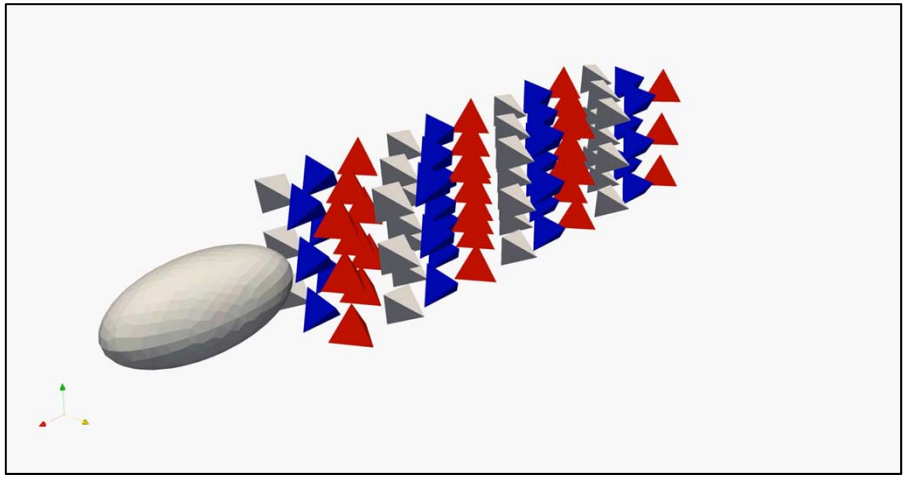
ENRICHMENT + BODY DYNAMICS + CONTACTS

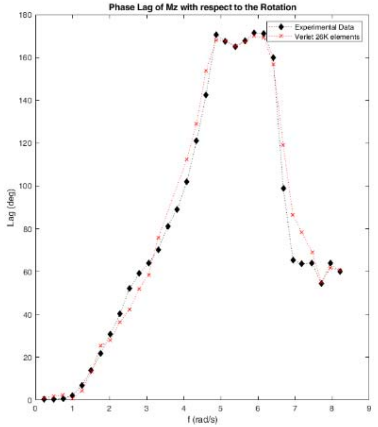
Fluid-Solid interface:

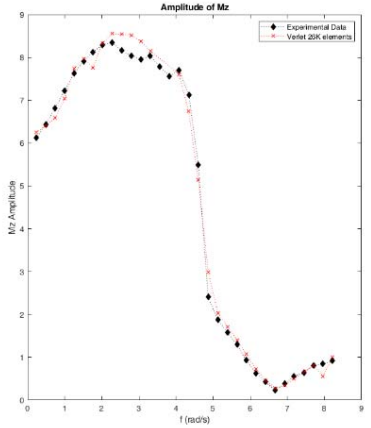


Water-Air interface:







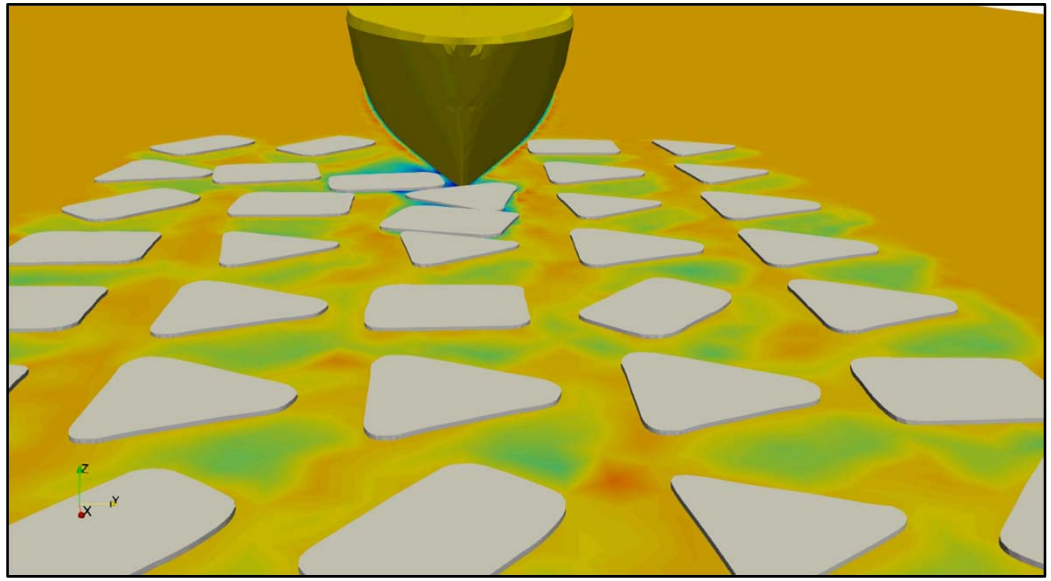


ONGOING AND FUTURE WORK RESEARCH LINES

ONGOING WORK

Ship navigation in ice conditions (NICESHIP project):

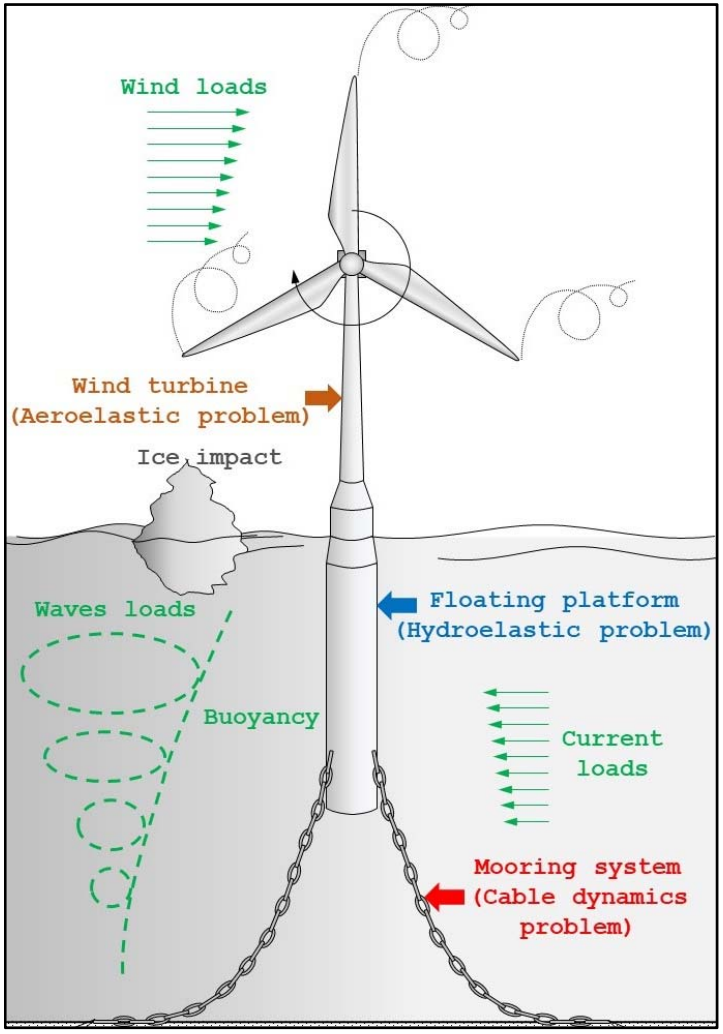
- Development of the SL-PFEM for high fidelity simulations of a ship navigating in ice conditions.
- Towing tank tests are being carried out at the ETSIN-UPM.
  - Paraffin wax is used to simulate ice.
  - The Hespérides hull will be analyzed.



ONGOING WORK

Simulation of Floating Offshore Wind Turbines (FOWT):

- Tayloring the SL-PFEM for high fidelity simulation of seakeeping hydrodynamics.
- Coupling the SL-PFEM with:
  - OpenFAST (Aerolasticity)
  - Structural solver (in house)
  - Mooring solver (in house).
- Development of Structural Reduced Order Models
- Development of Virtual twin for structural health monitoring.



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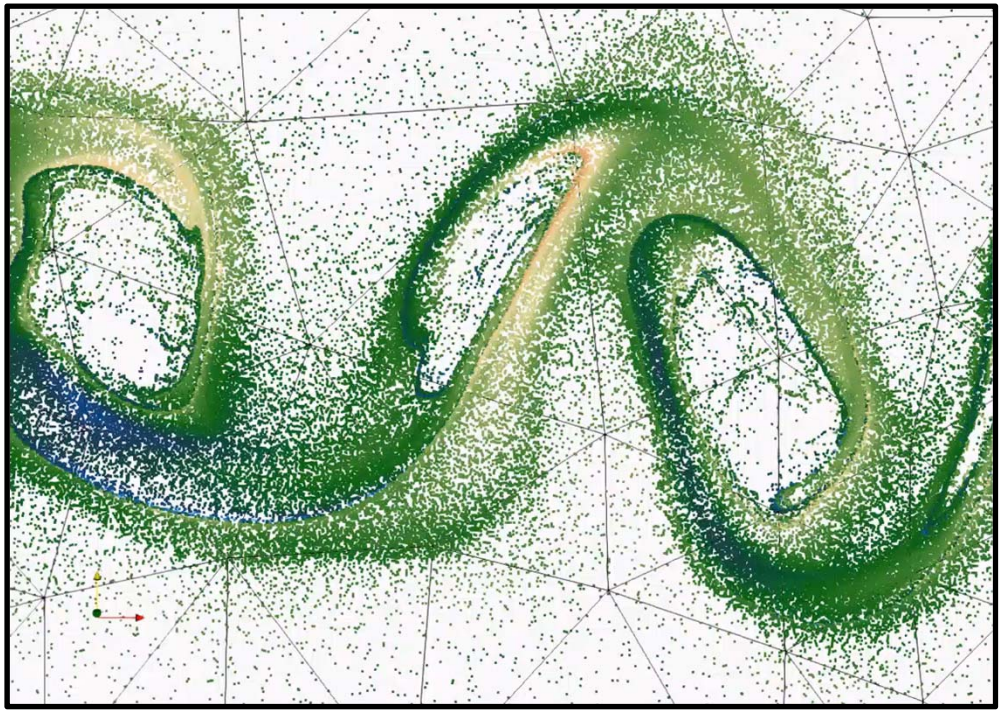
FUTURE WORK

FUTURE WORK

Future Work

- ✓ Resolve subgrid scales
  - ✓ Solve particle scale vorticity using a particle based vortex method.
- ✓ Lagrangian approach for turbulence modelling
- ✓ Create computational framework for Multiphysics problems.



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